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The soliton solutions of ϕ^6 field theory at finite temperature

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Abstract. The pair-cutoff real-time Green function method with the coherent state approach is used to investigate the (1+1)-dimensional ϕ^6 field theory. The topological and non-topological soliton solutions as well as their elementary excitation spectra at finite temperature are found. Two critical temperatures, one corresponding to the topological soliton disappearance and the other to the symmetry breaking restoration, are given.

1. Introduction

Recently considerable interest has been shown in the behaviour of the (1+1)-dimensional and (1+3)-dimensional ϕ^6 field theory at zero temperature and finite temperature. This is due to its wide applications in solid state physics and quantum field theory (Behera and Khare 1980, Muller and Schiemann 1986, Stevenson 1984, 1985, Stevenson and Rodit 1986, Barnes and Daniell 1984). In this paper we focus our attention on the soliton solutions and the phase structure of the (1+1)-dimensional ϕ^6 field theory, which may have three real vacua (as can be seen from figure 1) or two real vacua and a false vacuum (figure 2). This will give us more information about the restoration of the spontaneous symmetry breaking at critical temperature than ϕ^4 field theory or $\phi^3+\phi^4$ field theory. Using different methods, for example, the Gaussian effective potential approach (Roditi 1986) or the functional diagrammatic method (Babu Joseph and Kuriakose 1982), several authors made much effort to discuss the behaviour of the ϕ^6 field at finite temperature. So far, however, the topological and

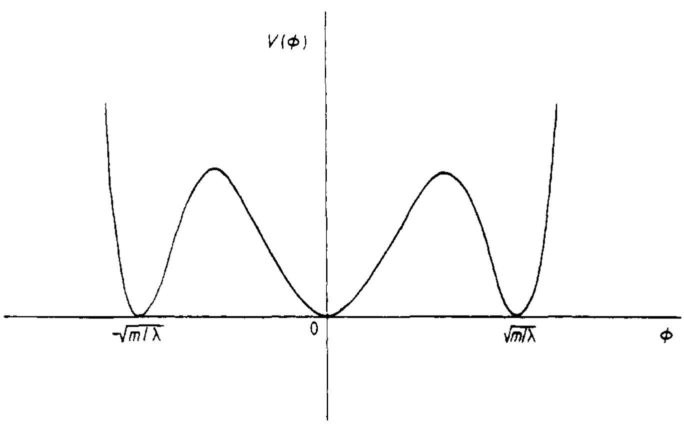

Figure 1. The classical potential of the ϕ^6 field at the special case $g^2=\lambda m$.

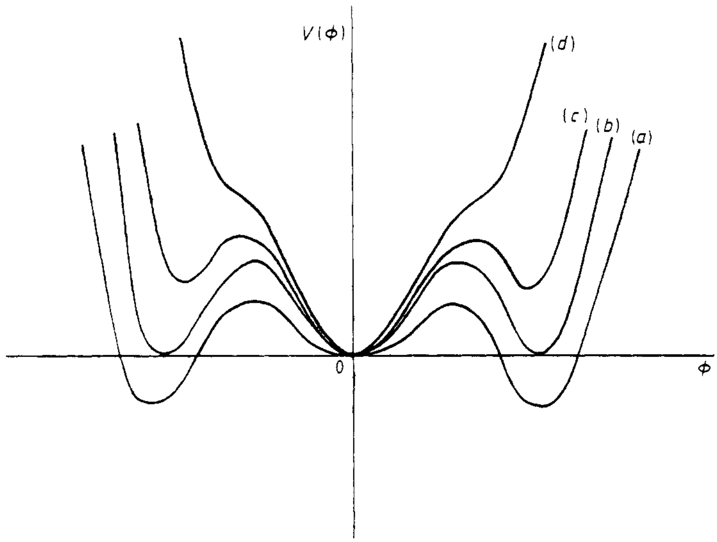

Figure 2. The effective potential of the ϕ^6 field at different temperatures: (a) $T=0$, (b) $T=T_{c_1}$, (c) $T_{c_1} \leq T \leq T_{c_2}$, ($d$) $T=T_{c_2}$.

non-topological soliton solutions as well as the elementary excitation spectra in a ϕ^6 field theory at zero or finite temperature have not been given.

In a series of previous papers, the method developed by one of the present authors and his co-workers had been widely used to investigate many physical systems, such as the ϕ^4 field theory (Su *et al* 1983), the $\phi^3+\phi^4$ field theory (Su and Gu 1986), $U(1)+\phi^4$ field theory, the Mohapatra–Senjanovic model (Su and Bi 1984a, b), the equation of state of nuclear matter (Su and Kuo 1987) and the phase transition of nuclear matter with Skyrme interactions (Su *et al* 1987). The construction of coherent states through the generalised Bogoliubov transformation and the pair-cutoff real-time Green functions play central roles in our method. We use this method to discuss the ϕ^6 field theory in the present paper, referring for the detail of this method to Su *et al* (1983).

The organisation of this paper is as follows. In § 2, we quantise and renormalise the Hamiltonian of the ϕ^6 model and find four types of solutions, of which two are topological and two are non-topological soliton solutions. In § 3, using the pair-cutoff real-time Green functions, we obtain the elementary excitations for all the cases at zero temperature. In § 4, we extend our calculations to finite temperature and develop two conceptions about critical temperatures, one of them corresponding to the disappearance of the topological soliton sector and the other to the conventional restoration of the spontaneous symmetry breaking. In § 5, we give a summary and discussion.

2. Hamiltonian and solutions

The Lagrangian density of a $(1+1)$-dimensional ϕ^6 field is

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m_b^2\phi^2 + g_b^2\phi^4 - \tfrac{1}{2}\lambda^2\phi^6 \tag{2.1}$$

where m_b, g_b, $\lambda > 0$, are bare coupling constants. The corresponding classical potential

$$V(\phi) = \tfrac{1}{2}m_b^2\phi^2 - g_b^2\phi^4 + \tfrac{1}{2}\lambda^2\phi^6 \tag{2.2}$$

has two absolute minima under the condition

$$g_b^2 > \lambda m_b \tag{2.3}$$

and three absolute minima under

$$g_b^2 = \lambda m_b . \tag{2.4}$$

The latter is just the case discussed by Babu Joseph and Kuriakose (1982). Performing the canonical quantisation

$$\phi(x, t) = \sum_k \frac{1}{(2L\omega_k)^{1/2}} (\hat{a}_k + \hat{a}^\dagger_{-k}) \exp(\mathrm{i}kx) \tag{2.5}$$

with

$$[\hat{a}_k(t), \hat{a}^\dagger_{k'}(t)] = \delta_{kk'} \tag{2.6}$$

$$\omega_k = (\mu^2 + \kappa^2)^{1/2} \tag{2.7}$$

and normal ordering procedure

$$\begin{aligned} \phi^2 &= :\phi^2: + A \\ \phi^4 &= :\phi^4: + 6A :\phi^2: + 3AB \\ \phi^6 &= :\phi^6: + 15A :\phi^4: + 90A^2 :\phi^2: + 45A^2 B \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} A &= \sum_k \frac{1}{2L\omega_k} = \frac{1}{2\pi} \ln \frac{2\Lambda}{\mu} \\ B &= \sum_k \frac{1}{\omega_k} \end{aligned} \tag{2.9}$$

we obtain the following Hamiltonian:

$$:H: = H_2 + H_4 + H_6 \tag{2.10}$$

$$H_2 = \sum_k \left[\frac{1}{2}\left(\frac{k^2}{\omega_k} + \omega_k \right) + \frac{m^2}{\omega_k} \right) \hat{a}^\dagger_k \hat{a}_k + \frac{1}{4}\left(\frac{k^2}{\omega_k} - \omega_k + \frac{m^2}{\omega_k} \right) (\hat{a}^\dagger_k \hat{a}^\dagger_{-k} + \hat{a}_k \hat{a}_{-k}) \right] \tag{2.11}$$

$$\begin{aligned} H_4 = -\frac{g^2}{4L} \sum_{k_1 k_2 k_3 k_4} & \frac{\delta_{(k_1+k_2+k_3+k_4),0}}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4})^{1/2}} [\hat{a}_{k_1}\hat{a}_{k_2}\hat{a}_{k_3}\hat{a}_{k_4} + \hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}^\dagger_{-k_3}\hat{a}^\dagger_{-k_4} \\ & + 4(\hat{a}^\dagger_{-k_1}\hat{a}_{k_2}\hat{a}_{k_3}\hat{a}_{k_4} + \hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}^\dagger_{-k_3}\hat{a}_{k_4}) + 6\hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}_{k_3}\hat{a}_{k_4}] \end{aligned} \tag{2.12}$$

$$\begin{aligned} H_6 = \frac{\lambda^2}{16L^2} \sum_{k_1 k_2 k_3 k_4 k_5 k_6} & \frac{\delta_{(k_1+k_2+k_3+k_4+k_5+k_6),0}}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4}\omega_{k_5}\omega_{k_6})^{1/2}} [\hat{a}_{k_1}\hat{a}_{k_2}\hat{a}_{k_3}\hat{a}_{k_4}\hat{a}_{k_5}\hat{a}_{k_6} \\ & + \hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}^\dagger_{-k_3}\hat{a}^\dagger_{-k_4}\hat{a}^\dagger_{-k_5}\hat{a}^\dagger_{-k_6} \\ & + 6(\hat{a}^\dagger_{-k_1}\hat{a}_{k_2}\hat{a}_{k_3}\hat{a}_{k_4}\hat{a}_{k_5}\hat{a}_{k_6} + \hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}^\dagger_{-k_3}\hat{a}^\dagger_{-k_4}\hat{a}^\dagger_{-k_5}\hat{a}_{k_6}) \\ & + 15(\hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}_{k_3}\hat{a}_{k_4}\hat{a}_{k_5}\hat{a}_{k_6} + \hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}^\dagger_{-k_3}\hat{a}^\dagger_{-k_4}\hat{a}_{k_5}\hat{a}_{k_6}) \\ & + 20\hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}^\dagger_{-k_3}\hat{a}_{k_4}\hat{a}_{k_5}\hat{a}_{k_6}] \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} m^2 &= m_b^2 - 12 g_b^2 A + 90\lambda^2 A^2 \\ g^2 &= g_b^2 - \tfrac{15}{2}\lambda^2 A \end{aligned} \tag{2.14}$$

are the renormalised parameters. However, λ does not change in this renormalisation. The equivalence of the normal ordering procedure and renormalisation was shown by Coleman (1975). Our notation and conventions follow those of Su *et al* (1983) and Su and Gu (1986) in this paper.

Performing the generalised Bogoliubov transformation

$$\hat{a}(k)=f(k)+\hat{c}(k) \tag{2.15}$$

to construct the coherent state configurations, where $f(k)$ is a c number which characterises the coherent state $|f\rangle$ and $c(k)$ is an operator which refers to the quantum fluctuation, and using the same variational procedure as Su *et al* (1983) to determine $f(k)$, we obtain

$$-\frac{d^2\tilde{y}}{du^2}+m^2\tilde{y}-16\pi g^2\tilde{y}^3+48\pi^2\lambda^2\tilde{y}^5=0 \tag{2.16}$$

where $\tilde{y}$ satisfies

$$y(p)=\frac{f(p)}{\omega_p^{1/2}}=\int_{-\infty}^{\infty}\tilde{y}(u)\exp(-ipu)\,du. \tag{2.17}$$

In fact, (2.16) describes the classical minima in Euclidean space with renormalised parameters. We will discuss its solutions below. The quantum fluctuation $c(k)$ will be discussed in the next section.

Equation (2.16) has four types of solution, as follows.

(a)

$$\begin{aligned}
&\tilde{y}(u)=0\\
&f(p)=0\\
&\langle\phi\rangle=0\\
&\langle{:}H{:}\rangle=U(\mathrm{a})=0.
\end{aligned} \tag{2.18}$$

This is a trivial solution corresponding to a false vacuum if $g^2>\lambda m$; but it becomes a true vacuum and has the same energy as solution (b) if $g^2=\lambda m$.

(b)

$$\begin{aligned}
&\tilde{y}=\pm\left(\frac{g^2}{6\pi\lambda^2}+\frac{(4g^4-3m^2\lambda^2)^{1/2}}{12\pi\lambda^2}\right)^{1/2}\\
&f(p)=\pm\left(\frac{2g^2+(4g^4-3m^2\lambda^2)^{1/2}}{3\lambda^2}\right)^{1/2}\omega_0^{1/2}\delta(p)\\
&\langle\phi\rangle=\pm\left(\frac{2g^2+(4g^4-3m^2\lambda^2)^{1/2}}{3\lambda^2}\right)^{1/2}\\
&\langle{:}H{:}\rangle=U(\mathrm{b})=\frac{(9m^2\lambda^2-8g^4)g^2}{27\lambda^4}+\frac{(3m^2\lambda^2-4g^4)(4g^4-3m^2\lambda^2)^{1/2}}{27\lambda^4}.
\end{aligned} \tag{2.19}$$

These solutions correspond to the spontaneous symmetry breaking vacua in which the boson condensation with zero momentum occurs. Here we have neglected two other solutions corresponding to the unstable maximum state. Obviously, when $g^2=m\lambda$, we get $\langle\phi\rangle = \pm(m/\lambda)^{1/2}$ and $\langle :H: \rangle = 0$ from (2.19). The vacuum is threefold degenerate; solutions (a) and (b) have the same energy in this condition.

(c)

$$\tilde{y}(u) = \pm \frac{m\,\mathrm{e}^{\pm mu}}{\{4\pi g^2\,\mathrm{e}^{\pm 2mu} + (2\pi m^2/k)[1+(g^4/m^4 - \lambda^2/m^2)\,\mathrm{e}^{\pm 4mu}]\}^{1/2}}$$

$$\langle\phi\rangle = \pm \frac{\sqrt{2k}m\,\mathrm{e}^{\pm mx}}{\{2g^2k\,\mathrm{e}^{\pm 2mx} + m^2[1+(g^4/m^4 - \lambda^2/m^2)\,\mathrm{e}^{\pm 4mx}]\}^{1/2}}$$

$$\begin{aligned}\langle :H: \rangle = U(\mathrm{c}) = {} & \frac{m\eta}{2(\alpha^2-4\beta)^{1/2}} \ln\left(\frac{\alpha+(\alpha^2-4\beta)^{1/2}}{\alpha-(\alpha^2-4\beta)^{1/2}}\right) + \frac{2g^2\eta^2}{m(\alpha^2-4\beta)} \\ & - \frac{g^2\alpha\eta}{m(\alpha^2-4\beta)^{1/2}} \ln\left(\frac{\alpha+(\alpha^2-4\beta)^{1/2}}{\alpha-(\alpha^2-4\beta)^{1/2}}\right) + \frac{\lambda^2\eta\alpha}{4m\beta(\alpha^2-4\beta)} - \frac{\lambda^2\eta^3\alpha^3}{4m\beta(\alpha^2-4\beta)^2} \\ & - \frac{\lambda^2\eta\alpha}{2m(\alpha^2-4\beta)^2} + \frac{\lambda^2\eta^3(\alpha^2+2\beta)}{2m(\alpha^2-4\beta)^{5/2}} \ln\left(\frac{\alpha+(\alpha^2-4\beta)^{1/2}}{\alpha-(\alpha^2-4\beta)^{1/2}}\right)\end{aligned} \tag{2.20}$$

where

$$\begin{aligned} & k = k_0 + (k_0 - k_0^2)^{1/2} \qquad k_0^2 = g^4/m^4 - \lambda^2/m^2 < 1 \\ & \alpha = 2[k_0 + (k_0 - k_0^2)^{1/2}]g^2/m^2 \qquad \beta = k_0^2 \qquad \eta = 2k. \end{aligned} \tag{2.21}$$

To obtain solutions (2.20), we used the method given by Su and Gu (1986) for the $\phi^3+\phi^4$ field, noting that (2.20) are the topological trivial solutions. They correspond to the one-dimensional motions from two turning points to the false vacuum. We had met these solutions in the $\phi^3+\phi^4$ field.

(d1) If $g^2 > \lambda m$, we obtain

$$\tilde{y}(u) = \frac{\pm\varphi_0\,\mathrm{sn}(\lambda[\varphi_1(\varphi_0^2-\varphi_2)]^{1/2}u)}{[4\pi(\varphi_0^2\,\mathrm{sn}(\lambda[\varphi_1(\varphi_0^2-\varphi_2)]^{1/2}u)+\varphi_2-\varphi_0^2)]^{1/2}}$$

$$\begin{aligned} & \langle\phi\rangle = \frac{\pm\varphi_0\,\mathrm{sn}(\lambda[\varphi_1(\varphi_0^2-\varphi_2)]^{1/2}\lambda)}{(\varphi_0^2\,\mathrm{sn}(\lambda[\varphi_1(\varphi_0^2-\varphi_2)]^{1/2}x)+\varphi_2-\varphi_0^2)^{1/2}} \\ & \langle :H: \rangle = U(\mathrm{d1}) = U(\mathrm{a})L + 2\int_{\varphi_0}^{\varphi_0} (m^2\varphi^2 - 2g^2\varphi^4 + \lambda^2\varphi^6 - 2U(\mathrm{a}))^{1/2}\,\mathrm{d}\varphi \end{aligned} \tag{2.22}$$

where $\mathrm{sn}(x)$ is the inverse elliptic function and

$$\begin{aligned} & \varphi_0^2 = [2g^2 + (4g^4 - 3m^2\lambda^2)^{1/2}]/3\lambda^2 \\ & \varphi_1 = \frac{\{4g^2 - (4g^4-3m^2\lambda^2)^{1/2} + g^2[48g^2 + 24(4g^4-3m^2\lambda^2)^{1/2}]^{1/2}\}}{6\lambda^2} \\ & \varphi_2 = \{4g^2 - (4g^4-3m^2\lambda^2)^{1/2} - g[48g^2+24(4g^4-3m^2\lambda^2)^{1/2}]^{1/2}\}/6\lambda^2. \end{aligned} \tag{2.23}$$

Noting that $\langle\phi(x \to \pm\infty)\rangle = \pm\phi_0$, (2.22) are the topological non-trivial solutions.

(d2) If $g^2=\lambda m$, we obtain

$$y^2(u)=\pm\left(\frac{m}{8\pi\lambda}\right)^{1/2}[1\pm\tanh(mu)]^{1/2}$$

$$\langle\phi(x)\rangle=\pm\left(\frac{m}{2\lambda}\right)^{1/2}[1\pm\tanh(mx)]^{1/2}$$

$$\begin{aligned}\langle{:}H{:}\rangle&=U(\mathrm{d}2)\\&=\left(\frac{m^2}{2\lambda}-\frac{m^2}{\lambda}+\frac{m^2}{2\lambda}\right)\ln\left(\frac{1+\mathrm{e}^{mL}}{1-\mathrm{e}^{-mL}}\right)-\frac{m^2}{4\lambda}\left(\frac{1}{1+\mathrm{e}^{2mL}}-\frac{1}{1+\mathrm{e}^{-2mL}}\right)\\&=\frac{m^2}{4\lambda}\qquad L\to\infty.\end{aligned}\tag{2.24}$$

Solutions of types (d1) and (d2) are the non-trivial topological kink and antikink solitons respectively. The particle can only move in two nearby vacua, namely $(0\sqrt{m/\lambda})$ or $(-\sqrt{m/\lambda},0)$, and since the velocity of the particle in vacuum is zero it cannot move to another new vacuum.

3. The real-time Green function at zero temperature

We define

$$G_1=\langle\!\langle\hat{c}_k|\hat{c}_k^{+}\rangle\!\rangle\qquad G_2=\langle\!\langle\hat{c}_{-k}^{+}|\hat{c}_k^{+}\rangle\!\rangle\tag{3.1}$$

as the zero-temperature Green functions, under normal and abnormal pair-cutoff approximation and obtain

$$G_1=\frac{1}{2\pi}\frac{E+\Omega_k}{E^2-(\Omega_k^2-\Delta_k^2)}\qquad G_2=-\frac{1}{2\pi}\frac{\Delta_k}{E^2-(\Omega_k^2-\Delta_k^2)}\tag{3.2}$$

in spectral representation, where

$$\begin{aligned}\Omega_k=&\frac{1}{2}\left(\frac{k^2}{\omega_k}+\omega_k\right)+\frac{m^2}{2\omega_k}-\frac{12g^2}{\omega_k L}\int\frac{\mathrm{d}k_1}{\omega_{k_1}}f(k_1)f(-k_1)\\&+\frac{15\lambda^2}{\pi\omega_k L}\int\frac{\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}k_4}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4})^{1/2}}\delta(k_1+k_2+k_3+k_4)f(k_1)f(k_2)f(k_3)f(k_4)\end{aligned}\tag{3.3}$$

$$\Delta_k=\Omega_k-\omega_k.\tag{3.4}$$

The elementary excitation spectrum can be obtained from the poles of G_1 and G_2

$$E_p^2=\Omega_p^2-\Delta_p^2.\tag{3.5}$$

In the uniform condensation phase which corresponds to solutions (b) of § 2, we obtain

$$E_p^2=p^2+m^2+\frac{5[2g^2+(4g^4-3m^2\lambda^2)^{1/2}]^2}{3\lambda^2}-\frac{4g^2[2g^2+(4g^4-3m^2\lambda^2)^{1/2}]}{\lambda^2}.\tag{3.6}$$

For the special case $g^2=\lambda m$, (3.6) becomes

$$E_p^2=p^2+4m^2.\tag{3.7}$$

In the topological trivial soliton case which corresponds to solutions (c) in § 2, we obtain

$$E_p^2 = p^2 + m^2 + \frac{15\lambda^2\eta^2}{m(4\beta-\alpha^2)}\frac{1}{L} + \frac{15\lambda^2\alpha\eta^2}{2mL(\alpha^2-4\beta)^{1/2}}\ln\left(\frac{\alpha+(\alpha^2-4\beta)^{1/2}}{\alpha-(\alpha^2-4\beta)^{1/2}}\right)$$
$$-\frac{12g^2\eta}{2mL(\alpha^2-4\beta)^{1/2}}\ln\left(\frac{\alpha+(\alpha^2-4\beta)^{1/2}}{\alpha-(\alpha^2-4\beta)^{1/2}}\right)$$
$$= p^2 + m^2 \qquad L\to\infty. \tag{3.8}$$

In the topological soliton case which corresponds to the solution (d1), we obtain

$$E_p^2 = p^2 + m^2 + \frac{15\lambda^2}{L}\int_{-\varphi_0}^{\varphi_0}\frac{\varphi^4\,\mathrm{d}\varphi}{(m^2\varphi^2-2g^2\varphi^4-\lambda^2\varphi^6-2U(\mathrm{a}))^{1/2}}$$
$$-\frac{12g^2}{L}\int_{-\varphi_0}^{\varphi_0}\frac{\varphi^2\,\mathrm{d}\varphi}{(m^2\varphi^2-2g^2\varphi^4+\lambda^2\varphi^6-2U(\mathrm{a}))^{1/2}}. \tag{3.9}$$

Equation (3.9) has two elliptic integrals that can be calculated numerically. However, it can be explicitly evaluated for the special case $g^2=\lambda m$, for which we obtain

$$E_p^2 = p^2 + m^2 + \frac{3m}{2L}\ln\left(\frac{1+\mathrm{e}^{mL}}{1+\mathrm{e}^{-mL}}\right) + \frac{15m}{2L}\left(\frac{1-\mathrm{e}^{mL}}{1+\mathrm{e}^{mL}}\right)$$
$$= p^2 + \tfrac{5}{2}m^2 \qquad L\to\infty. \tag{3.10}$$

4. The critical temperature and phase transition

After replacing the vacuum average by the ensemble average, we can extend our investigations to finite temperature. For the detail of this extension we refer to Su *et al* (1983) and Su and Gu (1986). Here we write down the main results only. We note that the Hamiltonian of the ϕ^6 field theory is more complicated than that of the ϕ^4 field theory since it has many three- and five-operator terms. After a little algebra, we can rewrite equation (2.16) as

$$-\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}u^2} + M^2\tilde{y} + 16\pi G^2\tilde{y}^3 + 48\pi^2\lambda^2\tilde{y}^5 = 0 \tag{4.1}$$

at finite temperature, where

$$M^2(\beta) = m^2 - 24g^2\nu + 180\lambda^2\nu^2 \tag{4.2}$$

$$G^2(\beta) = g^2 - 15\lambda^2\nu \tag{4.3}$$

$$\nu = \frac{1}{2\pi}\int_0^\infty \frac{\mathrm{d}k}{\omega_k}(\langle\hat{c}_k^\dagger\hat{c}_k\rangle + \langle\hat{c}_k^\dagger\hat{c}_{-k}^\dagger\rangle)$$
$$= \frac{1}{4\pi}\int_0^\infty\frac{\mathrm{d}k}{\omega_k}\left(\frac{(1+\omega_k/E_k)}{\mathrm{e}^{\beta E_k}-1} + \frac{(1-\omega_k/E_k)}{\mathrm{e}^{\beta E_k}-1}\right). \tag{4.4}$$

Comparing (2.16) and (4.1), we come to the conclusion that all formulae at zero temperature are still valid at finite temperature except the substitutions of M for m as well as G for g. This is the essential difference betwen ϕ^6 and ϕ^4 field theory, the latter demanding the substitution of M for m only.

Now we are in a position to discuss the critical temperatures and phase transitions. Obviously, equation (4.1) exhibits all the phase structures of our system, and the effective potential of (4.1) at finite temperature can be written as

$$V^{\beta}_{\mathrm{eff}}(\varphi)=\tfrac{1}{2}M^2\phi^2-G^2\phi^4+\tfrac{1}{2}\lambda^2\phi^6 \tag{4.5}$$

where M and G are functions of temperature as given by (4.2) and (4.3) and reduce to m and g at zero temperature. We can use this effective potential to discuss the phase transition of the ϕ^6 system as below.

4.1. *The critical temperature for topological soliton disappearance*

Noting that the topological solitons exist only in the region $G^2\geqslant\lambda M$ (when $G^2<\lambda M$, two degenerate true vacua will become false (figure 2) and the topological solitons will disappear) we find that

$$G^2=\lambda M \tag{4.6}$$

is a critical condition characterising the phase transition of the topological soliton of the ϕ^6 interaction. In the high-temperature region

$$\nu=T/4M+(1/4\pi)\ln(M/4\pi T)+\gamma/4\pi+\mathrm{O}(M^2/T^2). \tag{4.7}$$

Substituting (4.2), (4.3) and (4.7) into (4.6), we obtain the critical temperature

$$T_{c_1}\simeq 8(5m^2\lambda^2-4g^4)^{1/2}[g^2-(5m^2\lambda^2-4g^4)^{1/2}]/15\lambda^3. \tag{4.8}$$

4.2. *The critical temperature for breaking symmetry restoration*

As mentioned above, two false vacua and one true vacuum still exist when $T>T_{c_1}$ (figure 2). This means that the non-topological solitons still exist in this case. The condition in which two false vacua disappear is

$$\partial^2V^{\beta}_{\mathrm{eff}}/\partial\phi^2=M^2-16G^2\phi^2+15\lambda^2\phi^4\geqslant 0 \tag{4.9}$$

that is,

$$\Delta=12G^2-60\lambda^2M^2\leqslant 0 \tag{4.10}$$

while

$$\Delta=0 \tag{4.11}$$

is the critical condition where two false vacua become two inflection points and the non-topological solutions disappear. Substituting (4.2), (4.3) and (4.7) into (4.11), we obtain

$$T_{c_2}\simeq\left(\frac{4g^2-(10m^2\lambda^2-8g^4)^{1/2}}{15\lambda^2}\right)\left(\frac{15m^2\lambda^2-12g^4}{10}\right)^{1/2}. \tag{4.12}$$

All symmetry breaking will be restored where $T>T_{c_2}$.

5. Summary and discussion

In summary, we would like to point out the following.

(i) Using the pair-cutoff real-time Green function method with the coherent state approach, we obtain analytically the topological and non-topological solitons and elementary excitation spectra of a $(1+1)$-dimensional ϕ^6 field theory at zero temperature and finite temperature. Comparing our investigation with previous work we find that Babu Joseph *et al* (1982) discuss the special case $g^2=\lambda m$ which has three true vacua by a two-loop approximation only while Roditi (1986) tries to extend their Gaussian effective potential approach to finite temperature. The soliton solutions, the elementary excitation spectra, the phase structure and the critical temperature in the general case $g^2>\lambda m$ were not given in either of their papers.

(ii) In ϕ^4 field theory we can obtain only kink and antikink topological solitons since it has two true vacua. In $\phi^3+\phi^4$ field theory, we can obtain a non-topological soliton which corresponds to one-dimensional motion from the turning point to the false vacuum. We cannot obtain a non-topological soliton in ϕ^4 field theory or a topological soliton in $\phi^3+\phi^4$ field theory. But in ϕ^6 field theory, we can simultaneously obtain both topological and non-topological solitons in the case $g^2>\lambda m$. We can therefore say that the ϕ^6 field theory is more general than ϕ^4 or $\phi^3+\phi^4$ field theory in this sense. If one wishes to use a scalar field to construct a soliton bag (Friedberg and Lee 1977a, b, 1978, Goldflam and Wilets 1982, Bi *et al* 1986), then the ϕ^6 interaction may be a better choice than ϕ^4 and $\phi^3+\phi^4$ interactions.

(iii) We argue that there are two critical temperatures T_{c_1} and T_{c_2} in the pair-cutoff real-time Green function model, one corresponding to the topological soliton disappearance and the other to the symmetry breaking restoration. This is not surprising if one gives more detailed consideration to figure 2.

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